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On the Convergence of Galerkin Approximation Schemes for Second-Order Hyperbolic Equations in Energy and Negative Norms

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Abstract. Given certain semidiscrete and single step fully discrete Galerkin approximations to the solution of an initial-boundary value problem for a second-order hyperbolic equation, H^1 and L^2 error estimates are obtained. These estimates are valid simultaneously when the approximation to the initial data is taken to be the projection onto the approximating space with respect to the inner product which induces the energy norm that is naturally associated with the problem. The L^2-estimate is obtained as a by-product of the analysis of convergence in certain negative norms. Estimates are also obtained for the convergence of higher-order time derivatives in the presence of sufficiently smooth data.

1. Introduction. We consider the following initial-boundary value problem: Given a bounded domain $\Omega \subset \mathbf{R}^N$ with smooth boundary $\partial\Omega$, and $0 < t^* < \infty$, a function u is sought such that

$$(1.1)\qquad \begin{cases} D_t^2 u(t,x) + Lu(t,x) = 0 & \text{for } (t,x) \in (0,t^*] \times \Omega, \\ u(t,x) = 0 \quad \text{for } (t,x) \in (0,t) \times \partial\Omega, \\ u(0,x) = u_0(x), \qquad D_t u(0,x) = \dot{u}_0(x) & \text{for } x \in \Omega. \end{cases}$$

Here, u_0 and $\dot{u}_0$ are given functions and L denotes the second-order elliptic operator

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u,$$

with

$$a_{ij} = a_{ji} \in C^\infty(\overline{\Omega}), \qquad i,j = 1,2,\ldots,N, \quad a_0 \in C^\infty(\overline{\Omega}), \quad a_0 \geqslant 0 \quad \text{in } \Omega,$$

and

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqslant \alpha \sum_{i=1}^{N} \xi_i^2$$

for $x \in \overline{\Omega}$, all $(\xi_1, \xi_2, \ldots, \xi_N) \in \mathbf{R}^N$, α being some positive constant.

As in the paper [1] by Baker and Bramble on the approximation of (1.1), and the papers [2], [5], [6], [9] on Galerkin schemes for parabolic equations, we shall discuss the well-posedness of (1.1) and the convergence of approximation schemes within the framework of the spaces $\dot{H}^s(\Omega) \subset H^s(\Omega)$, $s \geqslant 0$, and their duals. Thus,

$$\dot{H}^0(\Omega) = L^2(\Omega),$$

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and

$$\dot{H}^s(\Omega) = \{v \in H^s(\Omega) : v = 0 \text{ and } L^j v = 0 \text{ on } \partial\Omega \text{ for } j < s/2\}$$

for $s > 0$. For $s < 0$, $\dot{H}^s(\Omega)$ denotes the dual, with respect to the L^2-inner product, of $\dot{H}^{-s}(\Omega)$. As shown in [6],

$$\dot{H}^s(\Omega) = \left\{ v \in L^2(\Omega) : \|v\|_s \equiv \left(\sum_{j=1}^{\infty} \left|(v, \varphi_j)\right|^2 \lambda_j^s \right)^{1/2} < \infty \right\}$$

for $s \geqslant 0$, where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots$ is the sequence of eigenvalues of the operator L with homogeneous Dirichlet boundary conditions, with the corresponding complete, orthonormal (in $L^2(\Omega)$) sequence of eigenfunctions $\{\varphi_j\}_{j=1}^{\infty}$. The norm $\|\cdot\|_s$ is equivalent to the usual Sobolev norm on $\dot{H}^s(\Omega)$, and on $L^2(\Omega)$ the dual norm induced by $\dot{H}^{-s}(\Omega)$ ($s \geqslant 0$) is equivalent to

$$\|v\|_{-s} = \left(\sum_{j=1}^{\infty} \left|(v, \varphi_j)\right|^2 \lambda_j^{-s} \right)^{1/2}.$$

$(\cdot, \cdot)$ will denote the duality between $\dot{H}^{-s}(\Omega)$ and $\dot{H}^s(\Omega)$ as well as the L^2-inner product, and $a(\cdot, \cdot)$ denotes the bilinear form associated with L, i.e.,

$$a(u, v) = \int_\Omega \left[\sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 uv \right] dx.$$

Let $T: \dot{H}^{-1}(\Omega) \to \dot{H}^1(\Omega)$ denote the solution operator defined by

$$a(Tf, \varphi) = (f, \varphi), \qquad \varphi \in \dot{H}^1(\Omega).$$

For $f \in L^2(\Omega)$, $Tf \in \dot{H}^2(\Omega)$ and can be represented as

$$Tf = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} (f, \varphi_j) \varphi_j.$$

One notes that, for $f \in L^2(\Omega)$,

$$\|f\|_{-s}^2 = (T^s f, f), \qquad s \geqslant 0. \tag{1.2}$$

When T is considered to be a linear operator in $L^2(\Omega)$, it is selfadjoint and positive definite [4], [9], so that

$$(v, w)_{-s} \equiv (T^s v, w), \qquad s \geqslant 0, \tag{1.3}$$

defines an inner product on $L^2(\Omega)$, and induces the norm $\|\cdot\|_{-s}$.

The initial-boundary value problem (1.1) may be viewed as an evolution equation for $U(t) \equiv [u(t), \dot{u}(t)]'$ ($'$ denotes the transpose) in the space $X \equiv \dot{H}^1(\Omega) \times L^2(\Omega)$;

$$\begin{cases} D_t U(t) + \Lambda U(t) = 0, \\ U(0) = U_0, \end{cases} \tag{1.4}$$

where

$$\Lambda = \begin{bmatrix} 0 & -I \\ L & 0 \end{bmatrix}, \tag{1.5}$$

$U_0 = [u_0, \dot{u}_0]'$, and $|||U|||_X \equiv |||U|||_0 \equiv \{\|u\|_1^2 + \|\dot{u}\|_0^2\}^{1/2}$ for $U = [u, \dot{u}]'$ is the 'energy' norm.

Using $|||U|||_q \equiv \{\|u\|_{q+1}^2 + \|\dot{u}\|_q^2\}^{1/2}$, $q \geqslant 0$, to denote the norm in $\dot{H}^{q+1}(\Omega) \times \dot{H}^q(\Omega)$, one observes that, for $U_0 \in \dot{H}^{q+1}(\Omega) \times \dot{H}^q(\Omega)$, $U(t) \in \dot{H}^{q+1}(\Omega) \times \dot{H}^q(\Omega)$, $t \in \mathbf{R}$, and that

$$|||U(t)|||_q = |||U(0)|||_q, \qquad t \in \mathbf{R}, q \geqslant 0. \tag{1.6}$$

This is easily deduced from the representation

$$u(t) = \sum_{j=1}^{\infty} \left[(u_0, \varphi_j)\cos\left(\sqrt{\lambda_j}\, t\right) + (\dot{u}_0, \varphi_j)\frac{\sin\left(\sqrt{\lambda_j}\, t\right)}{\sqrt{\lambda_j}} \right] \varphi_j. \tag{1.7}$$

For $q = 0$ (1.6) states the conservation of energy, for $q \geqslant 1$ it may be viewed as a regularity result pertaining to the solution to which we shall appeal frequently. For future reference let us also note that

$$|||U|||_q = |||\Lambda^q U|||_0 \tag{1.8}$$

for $U \in \dot{H}^{q+1}(\Omega) \times \dot{H}^q(\Omega)$. This follows readily from the characterization of the spaces $\dot{H}^q(\Omega)$ and the spectral representation of the norms $\|\cdot\|_q$, mentioned at the beginning, noting that

$$\Lambda^q = \begin{cases} (-1)^q \begin{bmatrix} L^{q/2} & 0 \\ 0 & L^{q/2} \end{bmatrix} & \text{for } q \text{ even}, \\ (-1)^{(q-1)/2} \begin{bmatrix} 0 & -L^{(q-1)/2} \\ L^{(q+1)/2} & 0 \end{bmatrix} & \text{for } q \text{ odd}. \end{cases}$$

The Galerkin formulation of (1.1) that is relevant to the approximation schemes to be considered in this paper results from

$$\left(D_t^2 u(t), \varphi\right) + a\left(u(t), \varphi\right) = 0 \quad \text{all } \varphi \in \dot{H}^1(\Omega),$$

with $u(t) \in \dot{H}^1(\Omega)$. As in the paper by Baker and Bramble [1], this may be cast as an evolution equation for $U(t) \in \dot{H}^1(\Omega) \times L^2(\Omega)$ with $D_t U(t) \in L^2(\Omega) \times \dot{H}^{-1}(\Omega)$;

$$\begin{cases} J D_t U(t) + U(t) = 0, & t > 0, \\ U(0) = U_0, \end{cases} \tag{1.9}$$

where

$$J \equiv \begin{bmatrix} 0 & T \\ -I & 0 \end{bmatrix}. \tag{1.9'}$$

One notes that J: $L^2(\Omega) \times \dot{H}^{-1}(\Omega) \to \dot{H}^1(\Omega) \times L^2(\Omega)$, so that (1.9) certainly makes sense. With $u(t)$ given by (1.7), $U(t) = [u(t), D_t u(t)]'$ is such a solution for $U_0 \in \dot{H}^1(\Omega) \times L^2(\Omega)$.

Parallel to the conservation of the 'positive' norms $|||\cdot|||_q$, $q \geqslant 0$, as expressed by (1.6), the negative norms defined by

$$|||U|||_{-p} \equiv \left\{\|u\|_{-(p-1)}^2 + \|\dot{u}\|_{-p}^2\right\}^{1/2}, \qquad p \geqslant 1,$$

so that

$$|||U|||_{-p} = |||U|||_{\dot{H}^{-(p-1)}(\Omega) \times \dot{H}^{-p}(\Omega)}$$

are also conserved:

$$|||U(t)|||_{-p} = |||U(0)|||_{-p}, \qquad t \in \mathbf{R}, \tag{1.10}$$

where $U(t)$ is the solution of (1.9). (1.10) easily follows from the representation (1.7). Nevertheless, we shall derive it in a way that will indicate to the reader the spirit in which we handle the Galerkin approximations.

We first note that $X = \dot{H}^1(\Omega) \times L^2(\Omega)$ is provided with the inner product denoted by

$$((U, V))_0 \equiv a(u, v) + (\dot{u}, \dot{v})$$

for $U = [u, \dot{u}]'$, $V = [v, \dot{v}]'$, and this inner product induces the energy norm $|||\cdot|||_0$. J, defined by (1.9′), is skew adjoint;

$$((JU, V))_0 = -((U, JV))_0, \qquad U, V \in X, \tag{1.11}$$

as is easily verified. In particular,

$$((JU, U))_0 = 0, \qquad U \in X. \tag{1.12}$$

Next, we observe that

$$|||U|||_{-p} = |||J^pU|||_0, \qquad p \geqslant 1. \tag{1.13}$$

Indeed,

$$|||J^pU|||_0^2 = ((J^pU, J^pU))_0 = (-1)^p((J^{2p}U, U))_0,$$

by the skew-adjointness of J ((1.11)), and

$$J^{2p} = (-1)^p\begin{bmatrix} T^p & 0 \\ 0 & T^p \end{bmatrix},$$

so that

$$\begin{aligned} |||J^pU|||_0^2 &= a(T^pu, u) + (T^p\dot{u}, \dot{u}) = (T^{p-1}u, u) + (T^p\dot{u}, \dot{u}) \\ &= \|u\|_{-(p-1)}^2 + \|\dot{u}\|_{-p}^2. \end{aligned}$$

Now, from (1.9) it follows that

$$J^{p+1}D_tU(t) + J^pU(t) = 0,$$

and

$$\left((J^{p+1}D_tU(t), J^pD_tU(t))\right)_0 + ((J^pU(t), J^pD_tU(t)))_0 = 0.$$

Due to the skew-adjointness of J ((1.12)), the first term falls away, and we obtain

$$\frac{1}{2}\frac{d}{dt}|||J^pU(t)|||_0^2 = 0,$$

and by (1.13)

$$\frac{d}{dt}|||U(t)|||_{-p}^2 = 0,$$

so that the conservation statement (1.10) holds for $p \geqslant 1$, as well as for $|||\cdot|||_0$.

We shall now describe the semidiscrete Galerkin scheme that will be considered in this paper. Let $S_h^r(\Omega) \subset \dot{H}^1(\Omega)$ be a finite dimensional subspace with the approximation property

$$\inf_{\varphi_h \in S_h^r(\Omega)} \left\{\|u - \varphi_h\|_0 + h\|u - \varphi_h\|_1\right\} \leqslant Ch^q\|u\|_q, \qquad 1 \leqslant q \leqslant r, \tag{1.14}$$

and $r \geqslant 2$.

The solution operator T_h: $\dot{H}^{-1}(\Omega) \to S_h^r(\Omega)$, corresponding to T, is defined by

(1.15) $$a(T_h f, \varphi_h) = (f, \varphi_h) \quad \text{for all } \varphi_h \in S_h^r(\Omega),$$

with f a given element of $\dot{H}^{-1}(\Omega)$. $u_h(t) \in S_h^r(\Omega)$, the Galerkin approximation to the solution $u(t)$ of (1.1) is sought as that function which satisfies

(1.16) $$\begin{cases} (D_t^2 u_h(t), \varphi_h) + a(u_h(t), \varphi_h) = 0, & t > 0, \varphi_h \in S_h^r(\Omega), \\ u_h(0) = u_{0,h} \in S_h^r(\Omega), & D_t u_h(0) = \dot{u}_{0,h} \in S_h^r(\Omega). \end{cases}$$

As in [1], (1.16) is cast in the form

(1.17) $$\begin{cases} J_h D_t U_h(t) + U_h(t) = 0, & t > 0, \\ U_h(0) = U_{0,h}, \end{cases}$$

where

$$U_h(t) = [u_h(t), \dot{u}_h(t)]', \qquad U_{0,h} = [u_{0,h}, \dot{u}_{0,h}]',$$

and

(1.18) $$J_h = \begin{bmatrix} 0 & T_h \\ -I & 0 \end{bmatrix},$$

parallel to (1.9), (1.9'). J_h, written as in (1.18), is an operator $L^2(\Omega) \times \dot{H}^{-1}(\Omega) \to S_h^r(\Omega) \times L^2(\Omega)$. Just as J is skew adjoint in $X = \dot{H}^1(\Omega) \times L^2(\Omega)$, equipped with the inner product $((\cdot, \cdot))_0$, J_h is skew adjoint in $S_h^r(\Omega) \times L^2(\Omega)$ equipped with the same inner product. Indeed, for $U = [u_h, \dot{u}]'$, $V = [v_h, \dot{v}]'$ in $S_h^r(\Omega) \times L^2(\Omega)$,

$$\begin{aligned} ((J_h U, V))_0 &= (([T_h \dot{u}, -u_h]', [v_h, \dot{v}]'))_0 = a(T_h \dot{u}, v_h) - (u_h, \dot{v}) \\ &= (\dot{u}, v_h) - (u_h, \dot{v}), \end{aligned}$$

and

$$\begin{aligned} ((U, J_h V))_0 &= (([u_h, \dot{u}]', [T_h \dot{v}, -v_h]'))_0 = a(u_h, T_h \dot{v}) - (\dot{u}, v_h) \\ &= a(T_h \dot{v}, u_h) - (\dot{u}, v_h) = (\dot{v}, u_h) - (\dot{u}, v_h), \end{aligned}$$

by the definition (1.15) of T_h and the symmetry of $a(\cdot, \cdot)$, so that

(1.19) $$((J_h U, V))_0 = -((U, J_h V))_0, \qquad U, V \in S_h^r(\Omega) \times L^2(\Omega),$$

and in particular

(1.20) $$((J_h U, U))_0 = 0$$

for $U \in S_h^r(\Omega) \times L^2(\Omega)$.

Conservation of energy is readily obtained from (1.17) by making use of (1.20):

$$((J_h D_t U_h(t), D_t U_h(t)))_0 + ((U_h(t), D_t U_h(t)))_0 = 0,$$

the first term vanishes by (1.20), and

$$\frac{1}{2} \frac{d}{dt} |||U_h(t)|||_0^2 = 0,$$

so that

(1.21) $$|||U_h(t)|||_0 = |||U_h(0)|||_0, \qquad t \in \mathbf{R}.$$

Furthermore, the discrete counterparts of the negative norms $|||\cdot|||_{-p}, p \geqslant 1$, are also conserved. If we adopt Thomée's definition and notation [9], the seminorm

$$(1.22) \qquad \|v\|_{-s,h} \equiv (T_h^s v, v)^{1/2}, \qquad s \geqslant 1,$$

is induced by

$$(v, w)_{-s,h} \equiv (T_h^s v, w),$$

and we define the seminorm

$$(1.23) \qquad |||V|||_{-p,h} = \left\{\|v\|^2_{-(p-1),h} + \|\dot{v}\|^2_{-p,h}\right\}^{1/2}, \qquad p \geqslant 1,$$

which is induced by the inner product

$$((V, W))_{-p,h} \equiv (T_h^{p-1} u, v) + (T_h^p \dot{u}, \dot{v})$$

for $V = [v, \dot{v}]'$, $W = [w, \dot{w}]'$.

Just as in the case of $|||\cdot|||_{-p}$, we note that

$$(1.24) \qquad |||J_h^p V_h|||_0 = |||V_h|||_{-p,h}, \qquad p \geqslant 1,$$

for $V_h \in S_h^r(\Omega) \times L^2(\Omega)$. Indeed

$$|||J_h^p V_h|||_0^2 = ((J_h^p V_h, J_h^p V_h))_0 = (-1)^p ((J_h^{2p} V_h, V_h))_0$$

by the skew-adjointness of J_h on $S_h^r(\Omega) \times L^2(\Omega)$ ((1.19)), and

$$J_h^{2p} = (-1)^p \begin{bmatrix} T_h^p & 0 \\ 0 & T_h^p \end{bmatrix},$$

so that

$$\begin{aligned} |||J_h^p V|||_0^2 &= a(T_h^p v_h, V_h) + (T_h^p \dot{v}, \dot{v}) = (T_h^{p-1} v_h, v_h) + (T_h^p \dot{v}, \dot{v}) \\ &= \|v_h\|^2_{-(p-1),h} + \|\dot{v}\|^2_{-p,h} \end{aligned}$$

for $V_h = [v_h, \dot{v}] \in S_h^r(\Omega) \times L^2(\Omega)$. We now go back to (1.17) and obtain

$$J_h^{p+1} D_t U_h(t) + J_h^p U_h(t) = 0,$$

so that

$$((J_h^{p+1} D_t U_h(t), J_h^p D_t U(t)))_0 + ((J_h^p U_h(t), J_h^p D_t U_h(t)))_0 = 0$$

and, making use of (1.20),

$$\frac{1}{2} \frac{d}{dt} |||J_h^p U_h(t)|||_0^2 = 0,$$

which, by (1.24), yields

$$(1.25) \qquad |||U_h(t)|||_{-p,h} = |||U_h(0)|||_{-p,h}, \qquad t \in \mathbf{R}.$$

Thus the solution $U_h(t)$ of the Galerkin equation (1.17) conserves the discrete negative seminorm $|||\cdot|||_{-p,h}, p \geqslant 1$.

Let us note that $((\cdot,\cdot))_{-1,h}$ coincides with the inner product $((\cdot,\cdot))$ utilized by Baker and Bramble in [1] in order to obtain L^2-estimates for $u_h(t) - u(t)$:

$$|||U_h(t) - U(t)|||^2_{-1,h} = \|u_h(t) - u(t)\|_0^2 + \|\dot{u}_h(t) - \dot{u}(t)\|^2_{-1,h}.$$

We shall carry out our convergence analysis simultaneously within the energy, i.e., $|||\cdot|||_0$-framework, and within the context of the negative norms $|||\cdot|||_{-p}$, the relationship of which to $|||\cdot|||_{-p,h}$ will be stated presently, and not only $|||\cdot|||_{-1,h}$. We shall choose $U_h(0)$ to be $\mathbf{P}_h U_0$, where $\mathbf{P}_h$: $\dot{H}^1(\Omega) \times L^2(\Omega) \to S_h^r(\Omega) \times S_h^r(\Omega)$ denotes the projection with respect to $((\cdot,\cdot))_0$. Thus

$$\mathbf{P}_h V = \left[P_h^1 v, P_h^0 \dot{v}\right]',$$

for $V = [v, \dot{v}]'$, where P_h^1 denotes the Ritz projection onto $S_h^r(\Omega)$ with respect to $a(\cdot,\cdot)$, and P_h^0 denotes the L^2-projection onto $S_h^r(\Omega)$, so that

$$a\left(P_h^1 v, \varphi_h\right) = a(v, \varphi_h), \qquad \varphi_h \in S_h^r(\Omega), \tag{1.26}$$

$$\left(P_h^0 v, \varphi_h\right) = (v, \varphi_h), \qquad \varphi_h \in S_h^r(\Omega), \tag{1.27}$$

Baker and Bramble choose $U_h(0) = [P_h^0 u_0, P_h^0 \dot{u}_0]$ and obtain optimal L^2-estimates for $u_h(t) - u(t)$, in [1]. In general it cannot be expected that energy estimates will be obtained with this choice of initial data. Thus, one may view the results of our paper as complementing those of [1], within the spirit of Thomée's paper [9] on negative norm estimates for Galerkin approximations to the solutions of parabolic equations. The author is greatly indebted to the works of all three authors.

Before we state our results more explicitly, we shall state the approximation-theoretic results that will be needed in the sequel. The background is available in the papers already referred to.

The following results are well known:

$$\left\|v - P_h^1 v\right\|_{-p} \leqslant Ch^{p+q}\|v\|_q, \qquad -1 \leqslant p \leqslant r-2,\ 1 \leqslant q \leqslant r. \tag{1.28}$$

$$\left\|v - P_h^0 v\right\|_{-p} \leqslant Ch^{p+q}\|v\|_q, \qquad 0 \leqslant p \leqslant r,\ 0 \leqslant q \leqslant r. \tag{1.29}$$

$$\left\|(T - T_h)f\right\|_{-p} \leqslant Ch^{p+q+2}\|f\|_q, \qquad -1 \leqslant p \leqslant r-2,\ -1 \leqslant q \leqslant r-2. \tag{1.30}$$

From (1.28), (1.29) and the definitions of the norms $|||\cdot|||_{-p}$, $|||\cdot|||_q$, one readily obtains

$$|||V - \mathbf{P}_h V|||_{-p} \leqslant Ch^{p+q-1}|||V|||_{q-1}, \qquad 0 \leqslant p \leqslant r-1,\ 1 \leqslant q \leqslant r. \tag{1.31}$$

From (1.30) and the definitions of J and J_h it follows that

$$|||(J - J_h)F|||_{-p} \leqslant Ch^{p+q-1}|||F|||_{q-2}, \qquad 0 \leqslant p \leqslant r-1,\ 1 \leqslant q \leqslant r. \tag{1.32}$$

We also need to clarify the relation between $|||\cdot|||_{-p}$ and $|||\cdot|||_{-p,h}$. In [9], Thomée proved the following result (Lemma 1 in that paper): For $0 \leqslant p \leqslant r$ and $v \in L^2(\Omega)$,

$$\|v\|_{-p,h} \leqslant C\{\|v\|_{-p} + h^p\|v\|_0\}, \tag{1.33}$$

$$\|v\|_{-p} \leqslant C\{\|v\|_{-p,h} + h^p\|v\|_0\}. \tag{1.34}$$

Parallel to (1.33) and (1.34), one obtains

$$\|v\|_{-(p-1),h} \leqslant C\{\|v\|_{-(p-1)} + h^p\|v\|_1\}, \tag{1.35}$$

$$\|v\|_{-(p-1)} \leqslant C\{\|v\|_{-(p-1),h} + h^p\|v\|_1\} \tag{1.36}$$

for $v \in \dot{H}^1(\Omega)$, $0 \leqslant p \leqslant r-1$. The proof is similar to that of Thomée's proof of (1.33) and (1.34), making use of (1.30) and the inequality

$$\|v\|_{-1} \leqslant C\{h^2\|v\|_1 + h^{-p}\|v\|_{-(p+1)}\}$$

instead of

$$\|v\|_{-2} \leqslant C\{h^2\|v\|_0 + h^{-(p-1)}\|v\|_{-(p+1)}\}$$

both of which are easily obtained from the spectral representations. From (1.33), (1.34), (1.35) and (1.36), it follows that

$$|||V|||_{-p,h} \leqslant C\{|||V|||_{-p} + h^p|||V|||_0\}, \tag{1.37}$$

$$|||V|||_{-p} \leqslant C\{|||V|||_{-p,h} + h^p|||V|||_0\} \tag{1.38}$$

for $V \in \dot{H}^1(\Omega) \times L^2(\Omega)$, $0 \leqslant p \leqslant r - 1$.

In Section 2 we shall discuss the convergence of semidiscrete approximations and prove the following:

THEOREM 1. *If* $U_0 \in \dot{H}^{q+1}(\Omega) \times \dot{H}^q(\Omega)$, $U_h(0) = \mathbf{P}_h U_0$, *for* $0 \leqslant t \leqslant t^*$,

$$|||U(t) - U_h(t)|||_{-p} \leqslant C(t^*)h^{p+q-1}|||U_0|||_q, \qquad 0 \leqslant p \leqslant r-1,\ 1 \leqslant q \leqslant r.$$

In particular, one has the energy estimate

$$|||U(t) - U_h(t)|||_0 \leqslant C(t^*)h^{r-1}|||U_0|||_r,$$

and the L^2-*estimate*

$$\|u(t) - u_h(t)\|_0 \leqslant C(t^*)h^r|||U_0|||_r.$$

The reader will observe that these estimates are valid for the choice $U_h(0) = [P_h^0 u_0, P_h^0 \dot{u}_0]'$ if $S_h^r(\Omega)$ satisfies the inverse property

$$\|\varphi_h\|_1 \leqslant Ch^{-1}\|\varphi_h\|$$

for all $\varphi_h \in S_h^r(\Omega)$.

In Section 3 we shall give estimates for fully discrete approximations corresponding to the class of rational approximations of the exponential labelled by Baker and Bramble [1] as Class *i*-I. Imposing the appropriate stability condition, as in [1], the reader may readily obtain the corresponding results for rational approximations of Class *i*-II.

In Section 4 we shall give estimates for the convergence of higher-order time derivatives of semidiscrete approximations, parallel to the results in the paper [3] by Baker and Dougalis. In order to obtain estimates for $|||D_t^s U(t) - D_t^s U_h(t)|||_{-p}$, $0 \leqslant p \leqslant r - 1$, we choose $U_h(0) = J_h^{s+1}\Lambda^{s+1}U_0$, $s \geqslant 1$. This is one of the choices considered by Baker and Dougalis. These authors had been aiming at L^∞-estimates for $(u(t) - u_h(t))$, and made use of estimates for $|||D_t^s U(t) - D_t^s U_h(t)|||_{-1,h}$. We do not duplicate their effort in the direction of L^∞-estimates and present our results concerning $|||D_t^s U(t) - D_t^s U_h(t)|||_{-p}$, $0 \leqslant p \leqslant r - 1$, as results which are of interest in their own right. Neither do we attempt to utilize our estimates in order to obtain other results parallel to those obtained in [5] and [9] for parabolic problems.

2. Convergence Estimates for Semidiscrete Approximations. We are comparing the solution $U(t)$ of the evolution equation (1.9) and the solution $U_h(t)$ of the corresponding equation (1.17), with $U_h(0) = \mathbf{P}_h U(0)$. We shall first establish the energy estimate, then the estimates in the discrete negative norms, and combining these results we obtain the principal result of this section, stated as Theorem 1 in the

Introduction. The energy estimate is of course classical [8], but we still choose to include the proof, which is in line with the overall approach of the paper, and which, in our opinion, has aesthetic appeal.

PROPOSITION 1. *If* $U(t)$ *is the solution of* (1.9), $U_h(t)$ *is the solution of* (1.17) *with* $U_h(0) = \mathbf{P}_h U_0$, *and* $U_0 \in \dot{H}^{q+1}(\Omega) \times \dot{H}^q(\Omega)$,

$$|||U(t) - U_h(t)|||_0 \leqslant C(t^*)h^{q-1}|||U_0|||_q, \qquad 1 \leqslant q \leqslant r, 0 \leqslant t \leqslant t^*. \tag{2.1}$$

(*As usual, C will denote a generic constant which may have a different meaning at different places.*)

Proof. To begin with, the case $q = 1$ is trivial, since

$$|||U(t)|||_0 = |||U_0|||_0,$$

and

$$|||U_h(t)|||_0 = |||\mathbf{P}_h U_0|||_0 \leqslant |||U_0|||_0,$$

by (1.6), (1.21) and the fact that $\mathbf{P}_h$ is the projection with respect to $((\cdot, \cdot))_0$. Therefore we need to consider $2 \leqslant q \leqslant r$. Writing

$$U(t) - U_h(t) = \big(U(t) - \mathbf{P}_h U(t)\big) + \big(\mathbf{P}_h U(t) - U_h(t)\big),$$

and noting ((1.31)) that

$$\begin{aligned}|||U(t) - \mathbf{P}_h U(t)|||_0 &\leqslant Ch^{q-1}|||U(t)|||_{q-1} = Ch^{q-1}|||U_0|||_{q-1},\\ &\leqslant Ch^{q-1}|||U_0|||_q,\end{aligned}$$

we shall have to prove

$$|||\mathbf{P}_h U(t) - U_h(t)|||_0 \leqslant Ch^{q-1}|||U_0|||_q, \tag{2.2}$$

$2 \leqslant q \leqslant r$, in order to establish (2.1). Since

$$JD_tU(t) + U(t) = 0, \qquad J_hD_tU(t) + U(t) = (J_h - J)D_tU(t),$$

we have

$$\begin{aligned}J_hD_t\mathbf{P}_hU(t) + \mathbf{P}_hU(t) = (J_h - J)D_tU(t) &+ J_h(\mathbf{P}_hD_tU(t) - D_tU(t))\\ &+ (\mathbf{P}_hU(t) - U(t)).\end{aligned}$$

Set

$$\begin{aligned}\rho_h(t) &= (J_h - J)D_tU(t) + J_h(\mathbf{P}_h - I)D_tU(t) + (\mathbf{P}_h - I)U(t)\\ &= (J - J_h)\Lambda U(t) + J_h(I - \mathbf{P}_h)\Lambda U(t) + (\mathbf{P}_h - I)U(t),\end{aligned} \tag{2.3}$$

by (1.4). Thus,

$$J_hD_t\mathbf{P}_hU(t) + \mathbf{P}_hU(t) = \rho_h(t), \qquad \mathbf{P}_hU(0) = \mathbf{P}_hU_0,$$

and

$$J_hD_tU_h(t) + U_h(t) = 0, \qquad U_h(0) = \mathbf{P}_hU_0,$$

so that with $E_h^*(t) \equiv \mathbf{P}_hU(t) - U_h(t)$,

$$J_hD_tE_h^*(t) + E_h^*(t) = \rho_h(t), \qquad E_h^*(0) = 0.$$

By forming the $((\cdot, \cdot))_0$-inner product with $D_tE_h^*(t)$,

$$((J_hD_tE_h(t), D_tE_h^*(t)))_0 + ((E_h^*(t), D_tE_h^*(t)))_0 = ((\rho_h(t), D_tE_h^*(t)))_0,$$

and noting that the first term on the left falls away due to the skew-adjointness of J_h ((1.20)) on $S_h^r(\Omega) \times S_h^r(\Omega)$, we obtain

$$\frac{1}{2}\frac{d}{dt}|||E_h^*(t)|||_0^2 = ((\rho_h(t), D_t E_h^*(t)))_0$$
$$= \frac{d}{dt}((\rho_h(t), E_h^*(t)))_0 - ((D_t\rho_h(t), E_h^*(t)))_0,$$

and

$$|||E_h^*(t)|||_0^2 = |||E_h^*(0)|||_0^2 + 2((\rho_h(t), E_h^*(t)))_0 - 2((\rho_h(0), E_h^*(0)))$$
$$-2\int_0^t ((D_t\rho_h(\tau), E_h^*(\tau)))_0\, d\tau.$$

Since $E_h^*(0) = 0$,

$$|||E_h^*(t)|||_0^2 = 2((\rho_h(t), E_h^*(t)))_0 - 2\int_0^t ((D_t\rho_h(\tau), E_h^*(\tau)))_0\, d\tau.$$

This implies, as in the proof of Theorem 2.1 in [1], that

$$\frac{3}{4}\sup_{0\leqslant t\leqslant t^*} |||E_h^*(t)|||_0^2 \leqslant 4\sup_{0\leqslant t\leqslant t^*} |||\rho_h(t)|||_0^2 + \frac{1}{4}\sup_{0\leqslant t\leqslant t^*} |||E_h^*(t)|||_0^2$$
$$+4t^*\int_0^{t^*} |||D_t\rho_h(\tau)|||_0^2\, d\tau,$$

and finally

$$\sup_{0\leqslant t\leqslant t^*} |||E_h^*(t)|||_0^2 \leqslant C\sup_{0\leqslant t\leqslant t^*}\left(|||\rho_h(t)|||_0^2 + (t^*)^2|||D_t\rho_h(t)|||_0^2\right). \tag{2.4}$$

We shall now estimate $|||\rho_h(t)|||_0$ and $|||D_t\rho_h(t)|||_0$. By (2.3) and (1.4),

$$D_t\rho_h(t) = (J_h - J)\Lambda^2 U(t) + J_h(\mathbf{P}_h - I)\Lambda^2 U(t) + (I - \mathbf{P}_h)\Lambda U(t).$$

Obviously, it is sufficient to estimate $|||D_t\rho_h(t)|||_0$. By (1.32), (1.8) and (1.6),

$$|||(J_h - J)\Lambda^2 U(t)|||_0 \leqslant Ch^{q-1}|||\Lambda^2 U(t)|||_{q-2}$$
$$= Ch^{q-1}|||U(t)|||_q = Ch^{q-1}|||U_0|||_q.$$

By (1.31)

$$|||(I - \mathbf{P}_h)\Lambda U(t)|||_0 \leqslant Ch^{q-1}|||\Lambda U(t)|||_{q-1}$$
$$= Ch^{q-1}|||U(t)|||_q = Ch^{q-1}|||U_0|||_q.$$

Now

$$|||J_h(\mathbf{P}_h - I)\Lambda^2 U(t)|||_0 = |||(\mathbf{P}_h - I)\Lambda^2 U(t)|||_{-1,h}$$
$$\leqslant C\left(|||(\mathbf{P}_h - I)\Lambda^2 U(t)|||_{-1} + h|||(\mathbf{P}_h - I)\Lambda^2 U(t)|||_0\right),$$

by (1.37). We have

$$|||(\mathbf{P}_h - I)\Lambda^2 U(t)|||_{-1} \leqslant Ch^{1+(q-1)-1}|||\Lambda^2 U(t)|||_{q-2} = Ch^{q-1}|||U_0|||_q,$$

due to (1.31), and

$$|||(\mathbf{P}_h - I)\Lambda^2 U(t)|||_0 \leqslant Ch^{(q-1)-1}|||\Lambda^2 U(t)|||_{q-2} = Ch^{q-2}|||U_0|||_q,$$

again by (1.31).

Combining the above inequalities, we obtain

$$|||J_h(\mathbf{P}_h - I)\Lambda^2 U(t)|||_0 \leqslant Ch^{q-1}|||U_0|||_q.$$

Therefore

$$|||D_t\rho_h(t)|||_0 \leqslant Ch^{q-1}|||U_0|||_q,$$

and similarly,

$$|||\rho_h(t)|||_0 \leqslant Ch^{q-1}|||U_0|||_{q-1},$$

so that

$$\sup_{0\leqslant t\leqslant t^*} |||E_h^*(t)|||_0 \leqslant C(t^*)h^{q-1}|||U_0|||_q,$$

and the proposition has been established.

PROPOSITION 2. *If* $U(t)$ *is the solution of* (1.9), $U_h(t)$ *is the solution of* (1.17) *with* $U_h(0) = \mathbf{P}_h U_0$, *and* $U_0 \in \dot{H}^{q+1}(\Omega) \times \dot{H}^q(\Omega)$, $0 \leqslant t \leqslant t^*$,

$$|||U(t) - U_h(t)|||_{-p,h} \leqslant C(t^*)h^{p+q-1}|||U_0|||_q, \qquad 1 \leqslant p \leqslant r-1, 1 \leqslant q \leqslant r. \tag{2.5}$$

Proof. We write again

$$J_h D_t U(t) + U(t) = (J_h - J)D_t U(t),$$

so that

$$J_h^{p+1}D_t U(t) + J_h^p U(t) = J_h^p(J_h - J)D_t U(t),$$

and

$$J_h^{p+1}D_t U_h(t) + J_h^p U_h(t) = 0, \tag{2.6}$$

so that, with $E_h(t) \equiv U(t) - U_h(t)$,

$$\begin{cases} J_h^{p+1}D_t E_h(t) + J_h^p E_h(t) = J_h^p(J_h - J)D_t U(t), \\ E_h(0) = U_0 - \mathbf{P}_h U_0. \end{cases} \tag{2.7}$$

We set

$$\sigma_h(t) = (J_h - J)D_t U(t), \tag{2.8}$$

form the $((\cdot,\cdot))_0$-inner product of (2.7) with $J_h^p D_t E_h(t)$, and obtain

$$\begin{aligned} &((J_h^{p+1}D_t E_h(t), J_h^p D_t E_h(t)))_0 + ((J_h^p E_h(t), J_h^p D_t E_h(t)))_0 \\ &\qquad = ((J_h^p\sigma_h(t), J_h^p D_t E_h(t)))_0. \end{aligned} \tag{2.9}$$

Since $J_h(X) \subset S_h^r(\Omega) \times L^2(\Omega)$, and J_h is skew adjoint on $S_h^r(\Omega) \times L^2(\Omega)$, the first term in (2.9) drops out, and we obtain

$$\frac{1}{2}\frac{d}{dt}|||J_h^p E_h(t)|||_0^2 = ((J_h^p\sigma_h(t), D_t J_h^p E_h(t)))_0. \tag{2.10}$$

From (2.10) we obtain, in exactly the same way as in the proof of Proposition 1,

$$\sup_{0\leqslant t\leqslant t^*} |||J_h^p E_h(t)|||_0 \leqslant C(t^*) \sup_{0\leqslant t\leqslant t^*} (|||J_h^p\sigma_h(t)|||_0 + |||J_h^p D_t\sigma_h(t)|||_0 + |||J_h^p E_h(0)|||_0).$$

By (1.24), this means that

$$(2.11)\quad \sup_{0\leqslant t\leqslant t^*} |||U(t) - U_h(t)|||_{-p,h} \leqslant C(t^*) \sup_{0\leqslant t\leqslant t^*} \left(|||\sigma_h(t)|||_{-p,h} + |||D_t\sigma_h(t)|||_{-p,h} + |||E_h(0)|||_{-p,h}\right).$$

In order to complete the proof of the proposition, we shall estimate $|||\sigma_h(t)|||_{-p,h}$, $|||D_t\sigma_h(t)|||_{-p,h}$ and $|||E_h(0)|||_{-p,h}$. Again it suffices to demonstrate the estimation of the last two terms. Now,

$$D_t\sigma_h(t) = (J_h - J)D_t^2U(t) = (J_h - J)\Lambda^2U(t).$$

Making use of (1.37),

$$|||D_t\sigma_h(t)|||_{-p,h} \leqslant C\left(|||(J - J_h)\Lambda^2U(t)|||_{-p} + h^p|||(J - J_h)\Lambda^2U(t)|||_0\right).$$

By (1.32)

$$|||(J - J_h)\Lambda^2U(t)|||_{-p} \leqslant Ch^{p+q-1}|||\Lambda^2U(t)|||_{q-2} = Ch^{p+q-1}|||U(t)|||_q = Ch^{p+q-1}|||U_0|||_q.$$

Again by (1.32)

$$|||(J - J_h)\Lambda^2U(t)|||_0 \leqslant Ch^{q-1}|||U_0|||_q.$$

We therefore have

$$|||D_t\sigma_h(t)|||_{-p,h} \leqslant Ch^{p+q-1}|||U_0|||_q.$$

As for $|||E_h(0)|||_{-p,h}$,

$$|||E_h(0)|||_{-p,h} = |||(I - \mathbf{P}_h)U_0|||_{-p,h} \leqslant C\left(|||(I - \mathbf{P}_h)U_0|||_{-p} + h^p|||(I - \mathbf{P}_h)U_0|||_0\right) \leqslant Ch^{p+q-1}|||U_0|||_{q-1} \leqslant Ch^{p+q-1}|||U_0|||_q$$

by (1.31) and (1.35), and the proposition has been established.

We can now immediately establish Theorem 1.

THEOREM 1. *If* $U(t)$ *is the solution of* (1.9), $U_h(t)$ *is the solution of* (1.17) *with* $U_h(0) = \mathbf{P}_hU_0$, *and* $U_0 \in \dot{H}^{q+1}(\Omega) \times \dot{H}^q(\Omega)$, $0 \leqslant t \leqslant t^*$,

$$|||U(t) - U_h(t)|||_{-p} \leqslant C(t^*)h^{p+q-1}|||U_0|||_q, \qquad 0 \leqslant p \leqslant r - 1,\ 1 \leqslant q \leqslant r.$$

Proof. By (1.38)

$$|||U(t) - U_h(t)|||_{-p} \leqslant C\left(|||U(t) - U_h(t)|||_{-p,h} + h^p|||U(t) - U_h(t)|||_0\right) \leqslant Ch^{p+q-1}|||U_0|||_q,$$

by Proposition 1 and Proposition 2.

Remark 1. The choice $U_h(0) = [P_h^0u_0, P_h^0\dot{u}_0]'$ leads to similar estimates if $S_h^r(\Omega)$ satisfies the 'inverse' assumption

$$\|\varphi_h\|_1 \leqslant Ch^{-1}\|\varphi_h\|_0, \qquad \varphi_h \in S_h^r(\Omega).$$

Remark 2. If U_0 is not smooth enough to be in $\dot{H}^2(\Omega) \times \dot{H}^1(\Omega)$, but is merely an element of, say, $X = \dot{H}^1(\Omega) \times L^2(\Omega)$, one can still make sense of negative norm estimates. Let us assume $\operatorname{supp} u_0 \subset\subset \Omega$, and $\operatorname{supp} \dot{u}_0 \subset\subset \Omega$. Set $K_h^* U_0 = [K_h^* u_0, K_h^* \dot{u}_0]'$, where K_h^* is a smoothing operator (as considered for example, in [4]) so that

$$|||K_h^* U_0|||_1 \leqslant Ch^{-1}|||U_0|||_0 \quad \text{and} \quad |||K_h^* U_0 - U_0|||_{-(r-1)} \leqslant Ch^{r-1}|||U_0|||_0.$$

Then it is easily seen from our estimates that the choice $U_h(0) = \mathbf{P}_h(K_h^* U_0)$ leads to

$$|||U(t) - U_h(t)|||_{-(r-1)} \leqslant Ch^{r-2}|||U_0|||_0.$$

Thus for $r > 2$, we have convergence in the sense of distributions, to the solution, which is a solution also in the sense of distributions. As opposed to the parabolic case, where nonsmooth initial data is smoothed out at $t > 0$, in the hyperbolic case such a result is all one can expect (over all of Ω) in the presence of nonsmooth data.

3. Convergence Estimates for Certain Fully Discrete Approximation Schemes. Let us denote

$$(3.1) \qquad I_h \equiv \text{Identity on } S_h^r(\Omega), \qquad L_h = \left(T_h|_{S_h^r(\Omega)}\right)^{-1}$$

(T_h is positive definite on $S_h^r(\Omega)$ [5]), so that

$$(3.2) \qquad \Lambda_h \equiv \begin{bmatrix} 0 & -I_h \\ L_h & 0 \end{bmatrix}$$

is $(J_h^*)^{-1}$, $J_h^* \equiv J_h|_{S_h^r \times S_h^r}$. We can then rewrite (1.17) as

$$(3.3) \qquad \begin{cases} D_t U_h(t) + \Lambda_h U_h(t) = 0, & t > 0, \\ U_h(0) = \mathbf{P}_h U_0, \end{cases}$$

so that

$$(3.4) \qquad U_h(t) = e^{-t\Lambda_h}\mathbf{P}_h U_0.$$

We shall consider rational functions $r(z)$ with the approximation property

$$(3.5) \qquad |r(iy) - e^{-iy}| \leqslant C|y|^{\nu+1}, \qquad |y| \leqslant \sigma,$$

for constants $C > 0$, $\nu > 0$, $\sigma > 0$, and which are of Class i-I [1]:

$$(3.6) \qquad |r(iy)| \leqslant 1 \quad \text{for all } y \in \mathbf{R}.$$

The fully discrete approximation $\{W^n\}_{n=0}^\infty \subset S_h^r(\Omega) \times S_h^r(\Omega)$ to the solution $U(t)$ of (1.9) is then defined by

$$(3.7) \qquad \begin{cases} W^{n+1} = r(k\Lambda_h)W^n, & n = 0, 1, 2, \ldots, \\ W^0 = \mathbf{P}_h U_0, \end{cases}$$

where $k > 0$ is the time step, so that

$$(3.8) \qquad W^n = r^n(k\Lambda_h)\mathbf{P}_h U_0$$

is to be compared with $U_h(t)$, $t = nk$ ((3.4)).

In preparation for the derivation of the error estimates, we shall first discuss the spectral representation of the relevant functions of J_h within the context of $((\cdot,\cdot))_0$, parallel to the discussion in [1] within the framework of $((\cdot,\cdot))_{-1,h}$.

Let X denote the complexification of $\dot{H}^1(\Omega) \times L^2(\Omega)$ as well, so that

$$((\Phi, \Psi))_0 = a(\varphi, \bar{\psi}) + (\dot{\varphi}, \bar{\dot{\psi}})$$

for $\Phi = [\varphi, \dot{\varphi}]'$, $\Psi = [\psi, \dot{\psi}]'$, with $^-$ denoting the complex conjugate. Let us denote by $\tilde{J}_h$ the restriction of J_h to the Hilbert space $S_h^r(\Omega) \times L^2(\Omega)$ (with $((\cdot, \cdot))_0$). It is readily observed that $\tilde{J}_h$ is skew adjoint, as in the real case, the kernel of $\tilde{J}_h$ is $\{0\} \times \text{Kernel}\, T_h$, and

$$(\text{Kernel}\, \tilde{J}_h)^{\perp} = S_h^r(\Omega) \times (L^2(\Omega) \ominus \text{Kernel}\, T_h),$$
$$L^2(\Omega) \ominus \text{Kernel}\, T_h = \text{Image}\, T_h = S_h^r(\Omega)$$

(T_h is selfadjoint in $L^2(\Omega)$, and is positive definite on $S_h^r(\Omega)$ [5]), so that one has

$$S_h^r(\Omega) \times L^2(\Omega) = (\text{Kernel}\, \tilde{J}_h) \oplus (S_h^r(\Omega) \times S_h^r(\Omega)). \tag{3.9}$$

As in [1], let $\{\mu_j^h\}_{j=1}^M$ denote the nonzero eigenvalues of T_h, and let $\{\psi_j^h\}_{j=1}^M$ be a corresponding sequence of eigenfunctions, orthonormal in $L^2(\Omega)$. Then, the sequence $\{\Phi_j^h\}_{j=-M}^M$ ($j \neq 0$) in $S_h^r(\Omega) \times S_h^r(\Omega)$ defined by

$$\Phi_j^h = \frac{1}{\sqrt{2}} \begin{bmatrix} -i(\mu_j^h)^{1/2} \psi_j^h \\ \psi_j^h \end{bmatrix}, \qquad \Phi_{-j}^h = \frac{1}{\sqrt{2}} \begin{bmatrix} i(\mu_j^h)^{1/2} \psi_j^h \\ \psi_j^h \end{bmatrix}, \qquad j = 1, 2, \ldots, M,$$

is easily seen to be a sequence of orthonormal (with respect to $((\cdot, \cdot))_0$) eigenfunctions for $\tilde{J}_h$, complete in $S_h^r(\Omega) \times S_h^r(\Omega)$, and corresponding to the eigenvalues $\eta_j = i(\mu_j^h)^{1/2}$, $\eta_{-j} = -i(\mu_j^h)^{1/2}$, $j = 1, 2, \ldots, M$, respectively.

Thus, for any $\Phi \in X$, and any function f, analytic in a neighborhood of the points $\{\eta_j^{-1}\}_{j=-M}^M$

$$\begin{aligned} f(\Lambda_h)\mathbf{P}_h\Phi &= \sum_{j=-M}^{M}{}' f(\eta_j^{-1})((\mathbf{P}_h\Phi, \Phi_j^h))_0 \Phi_j^h \\ &= \sum_{j=-M}^{M}{}' f(\eta_j^{-1})((\Phi, \Phi_j^h))_0 \Phi_j^h \end{aligned} \tag{3.10}$$

($'$ indicates that $j = 0$ is omitted), and for any $\Phi \in S_h^r(\Omega) \times L^2(\Omega)$ (in particular, for any $\Phi \in J_h(X)$),

$$J_h^l \Phi = \sum_{j=-M}^{M}{}' \eta_j^l ((\Phi, \Phi_j^h))_0 \Phi_j^h, \qquad l \geqslant 1. \tag{3.11}$$

As in [1], an essential step in the comparison of W^h and $U_h(nk)$ is the introduction of an auxiliary function

$$U_0^{(k)} = [u_0^{(k)}, \dot{u}_0^{(k)}]' \in \dot{H}^\infty(\Omega) \times \dot{H}^\infty(\Omega),$$

such that

$$|||U_0^{(k)}|||_{q+m} \leqslant k^{-m} |||U_0|||_q, \tag{3.12}$$

$$|||U_0 - U_0^{(k)}|||_{-p} \leqslant k^{q+p} |||U_0|||_q \tag{3.13}$$

for $m, p, q \geqslant 0$ (these follow from the definitions of the norms and the observations in [1]).

We are now ready to prove Theorem 2.

THEOREM 2. *Assume* $U_0 \in (\dot{H}^{q+1}(\Omega) \times \dot{H}^q(\Omega)) \cap (\dot{H}^{s+1}(\Omega) \times \dot{H}^s(\Omega))$. *For* $2 \leqslant q \leqslant r, 2 \leqslant s \leqslant \nu + 1, nk \leqslant t^*$,

$$|||W^n - U(nk)|||_0 \leqslant C(t^*)\big(h^{q-1}|||U_0|||_q + k^{s-1}|||U_0|||_s\big). \tag{3.14}$$

Proof. Due to Proposition 1, we need only prove that

$$|||W^n - U_h(nk)|||_0 \leqslant C\big(h^{q-1}|||U_0|||_q + k^{s-1}|||U_0|||_s\big). \tag{3.15}$$

By (3.4) and (3.8) this amounts to proving

$$|||(r^n(k\Lambda_h) - \exp^n(-k\Lambda_h))\mathbf{P}_h U_0|||_0 \leqslant C\big(h^{q-1}|||U_0|||_q + k^{s-1}|||U_0|||_s\big). \tag{3.16}$$

We introduce the auxiliary function $U_0^{(k)}$,

$$\begin{aligned}(r^n(k\Lambda_h) - \exp^n(-k\Lambda_h))\mathbf{P}_h U_0 &= (r^n(k\Lambda_h) - \exp^n(-k\Lambda_h))\mathbf{P}_h U_0^{(k)} \\ &\quad + (r^n(k\Lambda_h) - \exp^n(-k\Lambda_h))\mathbf{P}_h\big(U_0 - U_0^{(k)}\big),\end{aligned} \tag{3.17}$$

and estimate these terms separately.

By (3.10)

$$r^n(k\Lambda_h)\mathbf{P}_h\big(U_0 - U_0^{(k)}\big) = \sum_{j=-M}^{M}{}' r\big(k\eta_j^{-1}\big)\Big(\big(U_0 - U_0^{(k)}, \Phi_j^h\big)\Big)_0 \Phi_j^h,$$

so that, by (3.6) and (3.13),

$$\begin{aligned}|||r^n(k\Lambda_h)\mathbf{P}_h\big(U_0 - U_0^{(k)}\big)|||_0^2 &\leqslant \sum_{j=-M}^{M}{}' \Big|\Big(\big(U_0 - U_0^{(k)}, \Phi_j^h\big)\Big)_0\Big|^2 \\ &\leqslant |||U_0 - U_0^{(k)}|||_0^2 \leqslant k^{2s}|||U_0|||_s^2.\end{aligned} \tag{3.18}$$

We also have, by (1.21) and (3.13),

$$\begin{aligned}|||\exp^n(-k\Lambda_h)\mathbf{P}_h\big(U_0 - U_0^{(k)}\big)|||_0 &= |||\mathbf{P}_h\big(U_0 - U_0^{(k)}\big)|||_0 \\ &\leqslant |||U_0 - U_0^{(k)}|||_0 \leqslant k^s|||U_0|||_s.\end{aligned} \tag{3.19}$$

Thus

$$|||(r^n(k\Lambda_h) - \exp^n(-k\Lambda_h))\mathbf{P}_h\big(U_0 - U_0^{(k)}\big)|||_0 \leqslant 2k^s|||U_0|||_s, \tag{3.20}$$

and, in order to establish (3.16), we are left with the task of establishing the estimate

$$|||F_n(k\Lambda_h)\mathbf{P}_h U_0^{(k)}|||_0 \leqslant C\big(h^{q-1}|||U_0|||_q + k^{s-1}|||U_0|||_s\big), \tag{3.21}$$

$2 \leqslant q \leqslant r, 2 \leqslant s \leqslant \nu + 1$, where

$$F_n(z) \equiv r^n(z) - e^{-nz}. \tag{3.22}$$

As in [1] (and [2]), we write

$$U_0^{(k)} = \sum_{l=0}^{s} J_h^l(J - J_h)\Lambda^{l+1}U_0^{(k)} + J_h^{s+1}\Lambda^{s+1}U_0^{(k)}, \tag{3.23}$$

so that

$$\begin{aligned}\mathbf{P}_h U_0^{(k)} &= \mathbf{P}_h(J - J_h)\Lambda U_0^{(k)} + \mathbf{P}_h J_h(J - J_h)\Lambda^2 U_0^{(k)} \\ &\quad + \sum_{l=2}^{s} J_h^l(J - J_h)\Lambda^{l+1}U_0^{(k)} + J_h^{s+1}\Lambda^{s+1}U_0^{(k)}.\end{aligned} \tag{3.24}$$

We note that

$$(3.25) \qquad \mathbf{P}_h(J - J_h)Z = 0, \qquad Z \in \dot{H}^1(\Omega) \times L^2(\Omega).$$

Indeed, for $Z = [z, \dot{z}]'$,

$$\mathbf{P}_h(J - J_h)Z = \left[P_h^1(T - T_h)\dot{z}, 0\right] = \left[\left(P_h^1 T - T_h\right)\dot{z}, 0\right] = 0,$$

since $P_h^1 T = T_h$;

$$a\left(P_h^1 T\dot{z}, \varphi_h\right) = a(T\dot{z}, \varphi_h) = (\dot{z}, \varphi_h) = a(T_h\dot{z}, \varphi_h), \qquad \varphi_h \in S_h^r(\Omega).$$

Thus

$$(3.26) \qquad \begin{aligned} |||F_n(k\Lambda_h)\mathbf{P}_h U_0^{(k)}|||_0 \leqslant{} & |||F_n(k\Lambda_h)\mathbf{P}_h J_h(J - J_h)\Lambda^2 U_0^{(k)}|||_0 \\ & + \sum_{l=2}^{s} |||F_n(k\Lambda_h) J_h^l (J - J_h)\Lambda^{l+1} U_0^{(k)}|||_0 \\ & + |||F_n(k\Lambda_h) J_h^{s+1}\Lambda^{s+1} U^{(k)}|||_0. \end{aligned}$$

Now, as in the derivation of (3.18), for any Z,

$$(3.27) \qquad |||r^n(k\Lambda_h)\mathbf{P}_h Z|||_0 \leqslant |||Z|||_0,$$

and

$$(3.28) \qquad |||\exp^n(-k\Lambda_h)\mathbf{P}_h Z|||_0 \leqslant |||Z|||_0,$$

so that

$$\begin{aligned} & |||F_n(k\Lambda_h)\mathbf{P}_h J_h(J - J_h)\Lambda^2 U_0^{(k)}|||_0 \\ & \quad \leqslant 2|||J_h(J - J_h)\Lambda^2 U_0^{(k)}|||_0 = 2|||(J - J_h)\Lambda^2 U_0^{(k)}|||_{-1,h} \\ & \quad \leqslant C\left(|||(J - J_h)\Lambda^2 U_0^{(k)}|||_{-1} + h|||(J - J_h)\Lambda^2 U_0^{(k)}|||_0\right), \end{aligned}$$

by (1.37).

By (1.32)

$$\begin{aligned} |||(J - J_h)\Lambda^2 U_0^{(k)}|||_{-1} & \leqslant Ch^q|||\Lambda^2 U_0^{(k)}|||_{q-2} = Ch^q|||U_0^{(k)}|||_q \\ & \leqslant Ch^q|||U_0|||_q, \\ |||(J - J_h)\Lambda^2 U_0^{(k)}|||_0 & \leqslant Ch^{q-1}|||U_0|||_q, \end{aligned}$$

and we obtain

$$(3.29) \qquad |||F_n(k\Lambda_h)\mathbf{P}_h J_h(J - J_h)\Lambda^2 U_0^{(k)}|||_0 \leqslant Ch^q|||U_0|||_q.$$

In order to estimate

$$|||F_n(k\Lambda_h) J_h^l (J - J_h)\Lambda^{l+1} U_0^{(k)}|||_0, \qquad 2 \leqslant l \leqslant s,$$

we first note that

$$(3.30) \qquad |||F_n(k\Lambda_h) J_h^l Z|||_0 \leqslant C(t^*)k^{l-2}|||J_h Z|||_0,$$

for $2 \leqslant l \leqslant \nu + 2$, $t = nk \leqslant t^*$ (the proof of this statement is similar to that of Lemma 3.2 of [1]).

By (3.30)

$$(3.31) \qquad \begin{aligned} & |||F_h(k\Lambda_h) J_h^l (J - J_h)\Lambda^{l+1} U_0^{(k)}|||_0 \\ & \quad \leqslant Ck^{l-2}|||J_h(J - J_h)\Lambda^{l+1} U_0^{(k)}|||_0 = Ck^{l-2}|||(J - J_h)\Lambda^{l+1} U_0^{(k)}|||_{-1,h}. \end{aligned}$$

By (1.32), (1.37), (3.12),

$$\begin{aligned}
&|||(J - J_h)\Lambda^{l+1}U_0^{(k)}|||_{-1,h}\\
&\qquad \leqslant C\big(|||(J - J_h)\Lambda^{l+1}U_0^{(k)}|||_{-1} + h|||(J - J_h)\Lambda^{l+1}U_0^{(k)}|||_0\big),\\
&|||(J - J_h)\Lambda^{l+1}U_0^{(k)}|||_{-1} \leqslant Ch^{q-1}|||\Lambda^{l+1}U_0^{(k)}|||_{q-3} = Ch^{q-1}|||U_0^{(k)}|||_{q+(l-2)}\\
&\qquad \leqslant Ch^{q-1}\cdot k^{-(l-2)}|||U_0|||_q,\\
&|||(J - J_h)\Lambda^{l+1}U_0^{(k)}|||_0 \leqslant Ch^{q-2}|||\Lambda^{l+1}U_0^{(k)}|||_{q-3} \leqslant Ch^{q-2}k^{-(l-2)}|||U_0|||_q,
\end{aligned}$$

so that (3.31) yields

$$|||F_n(k\Lambda_h)J_h^l(J - J_h)\Lambda^{l+1}U_0^{(k)}|||_0 \leqslant Ch^{q-1}|||U_0|||_q, \qquad 2 \leqslant l \leqslant s. \tag{3.32}$$

Finally

$$|||F_n(k\Lambda_h)J_h^{s+1}\Lambda^{s+1}U_0^{(k)}|||_0 \leqslant Ck^{s-1}|||J_h\Lambda^{s+1}U_0^{(k)}|||_0 \tag{3.33}$$

by (3.30), and

$$|||J_h\Lambda^{s+1}U_0^{(k)}|||_0 \leqslant C\big(|||U_0|||_s + k^{-(s-1)}\cdot h^{q-1}|||U_0|||_q\big), \tag{3.34}$$

so that

$$|||F_n(k\Lambda_h)J_h^{s+1}\Lambda^{s+1}U_0^{(k)}|||_0 \leqslant C\big(k^{s-1}|||U_0|||_s + h^{q-1}|||U_0|||_q\big), \tag{3.35}$$

once (3.34) is established:

$$\begin{aligned}
|||J_h\Lambda^{s+1}U_0^{(k)}|||_0 &\leqslant |||(J_h - J)\Lambda^{s+1}U_0^{(k)}|||_0 + |||J\Lambda^{s+1}U_0^{(k)}|||_0\\
&= |||(J_h - J)\Lambda^{s+1}U_0^{(k)}|||_0 + |||U_0^{(k)}|||_s\\
&\leqslant Ch^{q-1}|||\Lambda^{s+1}U_0^{(k)}|||_{q-2} + |||U_0^{(k)}|||_s\\
&= Ch^{q-1}|||U_0^{(k)}|||_{q+(s-1)} + |||U_0^{(k)}|||_s\\
&\leqslant Ch^{q-1}\cdot k^{-(s-1)}|||U_0|||_q + |||U_0|||_s
\end{aligned}$$

by (1.32) and (3.12).

Combining (3.29), (3.32), (3.35), we obtain (3.21) and the theorem is established.

Having established the energy estimate for the fully discrete approximation, we shall consider it sufficient to give the following $|||\cdot|||_{-p}$-estimates, $1 \leqslant p \leqslant r-1$, which can be compared with the $|||\cdot|||_{-1,h}$-estimate of Baker and Bramble [1]:

THEOREM 3. *For* $2 \leqslant q \leqslant r$, $2 \leqslant s \leqslant \nu + 1$, $1 \leqslant p \leqslant r - 1$, $nk \leqslant t^*$,

$$|||W^n - U_h(nk)|||_{-p,h} \leqslant C(t^*)\big(h^{p+q-1}|||U_0|||_q + k^{s-1}|||U_0|||_{s-1}\big), \tag{3.36}$$

$$|||W^n - U_h(nk)|||_{-p} \leqslant C(t^*)\big(h^{p+q-1}|||U_0|||_q + (k^{s-1} + k^{s-2}h^p)|||U_0|||_{s-1}\big), \tag{3.37}$$

$$|||W^n - U_h(nk)|||_{-p} \leqslant C(t^*)\big(h^{p+q-1}|||U_0|||_q + k^{s-1}|||U_0|||_s\big). \tag{3.38}$$

Proof. Once (3.36) is established, (3.37) and (3.38) follow by utilizing the energy estimate of Theorem 2:

$$\begin{aligned}
|||W^n - U_h(nk)|||_{-p} &\leqslant C\big(|||W^n - U_h(nk)|||_{-p,h} + h^p|||W^n - U_h(nk)|||_0\big)\\
&\leqslant C\big(h^{p+q-1}|||U_0|||_q + k^{s-1}|||U_0|||_{s-1}\big) + Ch^p\big(h^{q-1}|||U_0|||_q + k^{s-2}|||U_0|||_{s-1}\big)\\
&= C\big(h^{p+q-1}|||U_0|||_q + (k^{s-1} + k^{s-2}h^p)|||U_0|||_{s-1}\big),
\end{aligned}$$

and similarly

$$|||W^n - U_h(nk)|||_{-p} \leqslant C\big(h^{p+q-1}|||U_0|||_q + (k^{s-1} + k^{s-1}h^p)|||U_0|||_s\big)$$
$$\leqslant C\big(h^{p+q-1}|||U_0|||_q + k^{s-1}|||U_0|||_s\big).$$

Thus, we need to prove (3.36). As in the proof of Theorem 2, (3.36) is established once we prove that

$$(3.39)\qquad |||F_n(k\Lambda_h)\mathbf{P}_h U_0^{(k)}|||_{-p,h} \leqslant C(t^*)\big(h^{p+q-1}|||U_0|||_q + k^{s-1}|||U_0|||_s\big).$$

Again

$$(3.40)\qquad |||F_n(k\Lambda_h)\mathbf{P}_h U_0^{(k)}|||_{-p,h} \leqslant |||F_n(k\Lambda_h)\mathbf{P}_h J_h(J-J_h)\Lambda^2 U_0^{(k)}|||_{-p,h}$$
$$+ \sum_{l=2}^{s} |||F_n(k\Lambda_h)J_h^l(J-J_h)\Lambda^{l+1}U_0^{(k)}|||_{-p,h}$$
$$+ |||F_n(k\Lambda_h)J_h^{s+1}\Lambda^{s+1}U_0^{(k)}|||_{-p,h}.$$

We note that

$$J_h\mathbf{P}_h J_h = J_h^2.$$

Indeed, for $Z = [z, \dot{z}]'$,

$$J_h\mathbf{P}_h J_h Z = J_h\mathbf{P}_h[T_h\dot{z}, -z] = J_h\big[T_h\dot{z}, -P_h^0 z\big]$$
$$= \big[-T_h P_h^0 z, -T_h\dot{z}\big] = [-T_h z, -T_h\dot{z}] = J_h^2 Z.$$

Therefore (3.40) reads

$$(3.41)\qquad |||F_n(k\Lambda_h)\mathbf{P}_h U_0^{(k)}|||_{-p,h} \leqslant \sum_{l=1}^{s} |||F_n(k\Lambda_h)J_n^{p+l}(J-J_h)\Lambda^{l+1}U_0^{(k)}|||_0$$
$$+ |||F_n(k\Lambda_h)J_h^{s+p+1}\Lambda^{s+1}U_0^{(k)}|||_0.$$

As in the proof of Theorem 2,

$$(3.42)\qquad |||F_n(k\Lambda_h)J_h^l J_h^p(J-J_h)\Lambda^{l+1}U_0^{(k)}|||_0 \leqslant Ck^{l-1}|||J_h^p(J-J_h)\Lambda^{l+1}U_0^{(k)}|||_0$$
$$\leqslant Ck^{l-1}\cdot h^{p+q-1}\cdot k^{-(l-1)}|||U_0|||_q$$
$$= Ch^{p+q-1}|||U_0|||_q.$$

Finally,

$$(3.43)\qquad |||F_n(k\Lambda_h)J_h^{s+p+1}\Lambda^{s+1}U_0^{(k)}|||_0 = |||F_n(k\Lambda_h)J_h^s J_h^{p+1}\Lambda^{s+1}U_0^{(k)}|||_0$$
$$\leqslant Ck^{s-1}|||J_h^{p+1}\Lambda^{s+1}U_0^{(k)}|||_0$$
$$\leqslant Ck^{s-1}\big(|||J_h^p J\Lambda^{s+1}U_0^{(k)}|||_0 + |||J_h^p(J_h - J)\Lambda^{s+1}U_0^{(k)}|||_0\big).$$

We then observe that

$$(3.44)\qquad |||J_h^p\Lambda^s U_0^{(k)}|||_0 = |||J_h^{p-1}(J_h\Lambda)\Lambda^{s-1}U_0^{(k)}|||_0$$
$$\leqslant C|||\Lambda^{s-1}U_0^{(k)}|||_0 \leqslant C|||U_0|||_{s-1},$$

since

$$J_h\Lambda = \begin{bmatrix} T_h L & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} P_h^1 & 0 \\ 0 & I \end{bmatrix}$$

is bounded.

Now,

$$|||J_h^p(J_h - J)\Lambda^{s+1}U_0^{(k)}|||_0 = |||(J_h - J)\Lambda^{s+1}U_0^{(k)}|||_{-p,h}$$
$$\leqslant C\big(|||(J_h - J)\Lambda^{s+1}U_0^{(k)}|||_{-p} + h^p|||(J_h - J)\Lambda^{s+1}U_0^{(k)}|||_0\big),$$

and

$$|||(J_h - J)\Lambda^{s+1}U_0^{(k)}|||_{-p} \leqslant Ch^{p+q-1}|||\Lambda^{s+1}U_0^{(k)}|||_{q-2} = Ch^{p+q-1}|||U_0^{(k)}|||_{q+(s-1)}$$
$$\leqslant Ch^{p+q-1}k^{-(s-1)}|||U_0|||_q,$$

and similarly

$$|||(J_h - J)\Lambda^{s+1}U_0^{(k)}|||_0 \leqslant Ch^{q-1}\cdot k^{-(s-1)}|||U_0|||_q,$$

so that

(3.45) $$|||J_h^p(J - J_h)\Lambda^{s+1}U_0^{(k)}|||_0 \leqslant Ch^{p+q-1}k^{-(s-1)}|||U_0|||_q.$$

From (3.43), (3.44) and (3.45) it follows that

(3.46) $$|||F_n(k\Lambda_h)J_h^{s+p+1}\Lambda^{s+1}U_0^{(k)}|||_0 \leqslant C\big(k^{s-1}|||U_0|||_{s-1} + h^{p+q-1}|||U_0|||_q\big),$$

and (3.41), (3.42) and (3.46) lead to (3.39), so that the theorem is established.

4. Estimates for the Higher-Order Time Derivatives of Semidiscrete Approximations. As we noted in the Introduction, our objective in this section is to complement the results in the paper by Baker and Dougalis [3] by obtaining energy and negative norm estimates for $D_t^sU(t) - D_t^sU_h(t)$, where $U(t)$ is the solution of (1.9) and $U_h(t)$ is the solution of (1.17) with $U_h(0) = J_h^{s+1}\Lambda^{s+1}U_0$, $s \geqslant 1$.

THEOREM 4. *Assume* $U_0 \in \dot{H}^{s+q+1}(\Omega) \times \dot{H}^{s+q}(\Omega)$, $s \geqslant 1$, $2 \leqslant q \leqslant r$, *and* $U_h(0) = J_h^{s+1}\Lambda^{s+1}U_0$. *Then*

(4.1) $$|||D_t^sU(t) - D_t^sU_h(t)|||_{-p} \leqslant Ch^{p+q-1}|||U_0|||_{s+q}$$

for $0 \leqslant p \leqslant r - 1$.

Proof. We shall again derive the energy estimate first. Since

$$JD_tU(t) + U(t) = 0, \qquad JD_t\big(D_t^sU(t)\big) + D_t^sU(t) = 0,$$

and

(4.2) $$D_tU(t) = -\Lambda U(t),$$

(4.3) $$\begin{cases} JD_t\Lambda^sU(t) + \Lambda^sU(t) = 0, \\ \Lambda^sU(0) = \Lambda^sU_0. \end{cases}$$

Similarly

(4.3′) $$\begin{cases} J_hD_t\Lambda_h^sU_h(t) + \Lambda_h^sU_h(t) = 0, \\ \Lambda_h^sU_h(0) = \Lambda_hJ_h^2\Lambda^{s+1}U_0, \end{cases}$$

since

$$\Lambda_h^sU_h(0) = \Lambda_h^sJ_h^{s+1}\Lambda^{s+1}U_0 = \Lambda_h\big(\Lambda_h^{s-1}J_h^{s-1}\big)J_h^2\Lambda^{s+1}U_0.$$

We shall write

$$(4.4)\qquad \begin{aligned} D_t^s U_h(t) - D_t^s U_h(t) &= (-1)^s\left(\Lambda^s U(t) - \Lambda_h^s U_h(t)\right) \\ &= (-1)^s\left(\Lambda^s U(t) - \Lambda_h J_h^2 \Lambda^{s+1} U(t)\right) \\ &\quad + (-1)^s\left(\Lambda_h J_h^2 \Lambda^{s+1} U(t) - \Lambda_h^s U_h(t)\right) \\ &= (-1)^s\left(E_h^{**}(t) + E_h^*(t)\right), \end{aligned}$$

and estimate $E_h^{**}(t)$ and $E_h^*(t)$ separately. Both estimates rely upon the following: For $Z = [z, \dot{z}]'$, $1 \leqslant q \leqslant r$,

$$(4.5)\qquad |||(J - \Lambda_h J_h^2) Z|||_{-p} \leqslant C h^{p+q-1} |||Z|||_{q-2}.$$

This follows readily from (1.29), (1.30) and the expression (obtained from the definitions and the identity $P_h^0 = L_h T_h$)

$$J - \Lambda_h J_h^2 = \begin{bmatrix} 0 & T - T_h \\ P_h^0 - I & 0 \end{bmatrix}.$$

By (4.5)

$$(4.6)\qquad \begin{aligned} |||E_h^{**}(t)|||_0 &= |||\Lambda^s U(t) - \Lambda_h J_h^2 \Lambda^{s+1} U(t)|||_0 = |||(J - \Lambda_h J_h^2)\Lambda^{s+1} U(t)|||_0 \\ &\leqslant C h^{q-1} |||\Lambda^{s+1} U(t)|||_{q-2} = C h^{q-1} |||U(t)|||_{s+q-1} \\ &= C h^{q-1} |||U_0|||_{s+q-1}. \end{aligned}$$

In order to estimate $|||E_h^*(t)|||_0$, we obtain from (4.3)

$$J_h D_t \Lambda^s U(t) + \Lambda^s U(t) = (J_h - J) D_t \Lambda^s U(t),$$

$$(4.7)\qquad \begin{aligned} &J_h D_t\left(\Lambda_h J_h^2 \Lambda^{s+1} U(t)\right) + \Lambda_h J_h^2 \Lambda^{s+1} U(t) \\ &\quad = (J_h - J) D_t \Lambda^s U(t) + J_h D_t\left(\Lambda_h J_h^2 \Lambda^{s+1} U(t) - \Lambda^s U(t)\right) \\ &\qquad + \left(\Lambda_h J_h^2 \Lambda^{s+1} U(t) - \Lambda^s U(t)\right) \\ &\quad \equiv \tilde{\rho}_h(t). \end{aligned}$$

From (4.7) and (4.3′)

$$(4.8)\qquad \begin{cases} J_h D_t E_h^*(t) + E_h^*(t) = \tilde{\rho}_h(t) \\ E_h^*(0) = 0. \end{cases}$$

Just as in the proof of Proposition 1, (4.8) leads to

$$(4.9)\qquad |||E_h^*(t)|||_0 \leqslant C(t^*) \sup_{0 \leqslant t \leqslant t^*} \{|||\tilde{\rho}_h(t)|||_0 + |||D_t \tilde{\rho}_h(t)|||_0\}$$

for $0 \leqslant t \leqslant t^*$, and again it suffices to display the estimation of $|||D_t \tilde{\rho}_h(t)|||_0$: By (4.2)

$$(4.10)\qquad \begin{aligned} D_t \tilde{\rho}_h(t) &= (J_h - J)\Lambda^{s+2} U(t) + J_h\left(\Lambda_h J_h^2 \Lambda^{s+3} U(t) - \Lambda^{s+2} U(t)\right) \\ &\quad + \left(\Lambda^{s+1} U(t) - \Lambda_h J_h^2 \Lambda^{s+2} U(t)\right). \end{aligned}$$

By (1.32),

$$(4.11)\qquad \begin{aligned} |||(J_h - J)\Lambda^{s+2} U(t)|||_0 &\leqslant C h^{q-1} |||\Lambda^{s+2} U(t)|||_{q-2} \\ &= C h^{q-1} |||U(t)|||_{s+q} = C h^{q-1} |||U_0|||_{s+q}. \end{aligned}$$

By (4.5)

$$(4.12)\qquad \begin{aligned} |||\Lambda^{s+1}U(t) - \Lambda_h J_h^2\Lambda^{s+2}U(t)|||_0 &= |||(J - \Lambda_h J_h^2)\Lambda^{s+2}U(t)|||_0 \\ &\leq Ch^{q-1}|||\Lambda^{s+2}U(t)|||_{q-2} \\ &= Ch^{q-1}|||U(t)|||_{s+q} = Ch^{q-1}|||U_0|||_{s+q}, \end{aligned}$$

and by (4.5), (1.37),

$$(4.13)\quad \begin{aligned} |||J_h(\Lambda_h J_h^2 - J)\Lambda^{s+3}U(t)|||_0 &= |||(\Lambda_h J_h^2 - J)\Lambda^{s+3}U(t)|||_{-1,h} \\ &\leq C\left(|||(\Lambda_h J_h^2 - J)\Lambda^{s+3}U(t)|||_{-1} + h|||(\Lambda_h J_h^2 - J)\Lambda^{s+3}U(t)|||_0\right) \\ &\leq Ch^{q-1}|||\Lambda^{s+3}U(t)|||_{q-3} = Ch^{q-1}|||U(t)|||_{s+q} = Ch^{q-1}|||U_0|||_{s+q}. \end{aligned}$$

(4.10), (4.11), (4.12) and (4.13) lead to the estimate

$$(4.14)\qquad |||D_t\tilde{\rho}_h(t)|||_0 \leq Ch^{q-1}|||U_0|||_{s+q}.$$

Similarly

$$(4.15)\qquad |||\tilde{\rho}_h(t)|||_0 \leq Ch^{q-1}|||U_0|||_{s+q-1}.$$

By (4.9), (4.14), (4.15),

$$(4.16)\qquad |||E_h^*(t)|||_0 \leq C(t^*)h^{q-1}|||U_0|||_{s+q}, \qquad 0 \leq t \leq t^*,$$

and (4.4), (4.6), (4.16) yield the energy estimate

$$(4.17)\quad |||D_t^sU(t) - D_t^sU_h(t)|||_0 \leq C(t^*)h^{q-1}|||U_0|||_{s+q}, \qquad 0 \leq t \leq t^*, 2 \leq q \leq r.$$

In order to establish the negative norm estimates, due to (1.38) and (4.17), it suffices to establish that

$$(4.18)\quad |||D_t^sU(t) - D_t^sU_h(t)|||_{-p,h} \leq C(t^*)h^{p+q-1}|||U_0|||_{s+q}, \qquad 0 \leq t \leq t^*, 1 \leq p \leq r-1.$$

Since

$$JD_t\Lambda^sU(t) + \Lambda^sU(t) = 0,$$
$$J_hD_t\Lambda^sU(t) + \Lambda^sU(t) = (J_h - J)D_t\Lambda^sU(t),$$

we have

$$J_h^{p+1}D_t\Lambda^sU(t) + J_h^p\Lambda^sU(t) = J_h^p(J_h - J)D_t\Lambda^sU(t),$$

and

$$J_h^{p+1}D_t\Lambda_h^sU_h(t) + J_h^p\Lambda_h^sU_h(t) = 0,$$

so that

$$(4.19)\qquad J_h^{p+1}D_tE_h(t) + J_h^pE_h(t) = J_h^p\tilde{\sigma}_h(t),$$

where

$$(4.20)\qquad E_h(t) \equiv \Lambda^sU(t) - \Lambda_h^sU_h(t),$$

$$(4.21)\qquad \tilde{\sigma}_h(t) \equiv (J_h - J)D_t\Lambda^sU(t).$$

Just as in the proof of Proposition 2, (4.19) leads to the estimate

$$(4.22)\quad |||E_h(t)|||_{-p,h} \leq C(t^*)\left\{|||E_h(0)|||_{-p,h} + \sup_{0\leq t\leq t^*}\left(|||\tilde{\sigma}_h(t)|||_{-p,h} + |||D_t\tilde{\sigma}_h(t)|||_{-p,h}\right)\right\}.$$

Now,

$$E_h(0) = \Lambda^s U_0 - \Lambda_h^s J_h^{s+1} \Lambda^{s+1} U_0 = \left(J - \Lambda_h J_h^2\right) \Lambda^{s+1} U_0,$$

so that, by (1.37) and (4.5),

$$\begin{aligned}(4.23)\quad |||E_h(0)|||_{-p,h} &\leqslant C\left(|||\left(J - \Lambda_h J_h^2\right)\Lambda^{s+1}U_0|||_{-p} + h^p|||\left(J - \Lambda_h J_h^2\right)\Lambda^{s+1}U_0|||_0\right) \\ &\leqslant Ch^{p+q-1}|||\Lambda^{s+1}U_0|||_{q-2} = Ch^{p+q-1}|||U_0|||_{s+q-1}.\end{aligned}$$

As for $|||D_t\tilde{\sigma}_h(t)|||_{-p,h}$, we have, using (1.37) and (1.32),

$$\begin{aligned}(4.24)\quad & |||(J_h - J)\Lambda^{s+2}U(t)|||_{-p,h} \\ &\leqslant C\left(|||(J_h - J)\Lambda^{s+2}U(t)|||_{-p} + h^p|||(J_h - J)\Lambda^{s+2}U(t)|||_0\right) \\ &\leqslant Ch^{p+q-1}|||\Lambda^{s+2}U(t)|||_{q-2} = Ch^{p+q-1}|||U(t)|||_{s+q} \\ &= Ch^{p+q-1}|||U_0|||_{s+q}.\end{aligned}$$

Similarly,

$$(4.25)\qquad |||\tilde{\sigma}_h(t)|||_{-p,h} \leqslant Ch^{p+q-1}|||U_0|||_{s+q-1}.$$

(4.22), (4.23), (4.24) and (4.25) lead to (4.18), and the theorem is established.

5. Concluding Remarks. Even though we have examined a specific case, it is evident that the approach of the paper is relevant to Galerkin approximations of equations in the form

$$(5.1)\qquad D_t^2 v(t) + Av(t) = 0,$$

where A is a positive definite selfadjoint operator which may result from a plate problem or a problem in three-dimensional elasticity. Formally, (5.1) leads to the evolution equation

$$D_t\begin{bmatrix} v(t) \\ \dot{v}(t) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix}\begin{bmatrix} v(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is Hamiltonian with energy

$$\left|\left|\left|\begin{matrix} v \\ \dot{v} \end{matrix}\right|\right|\right| = \left\{(Av, v) + \|\dot{v}\|_0^2\right\}^{1/2}$$

[7], and it is this structure that we have exploited in our discussion of our specific case.

It might also be of interest to apply our approach to the nonhomogeneous equation

$$D_t^2 v(t) + Av(t) = f(t),$$

and obtain convergence results for nonsmooth data in terms of the negative norms (cf. Remark 2).

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